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# SOME REMARKS ON MAXIMAL SUBGROUPS OF FINITE CLASSICAL GROUPS

KAY MAGAARD

ABSTRACT. The subgroup structure of the finite classical groups has long been the subject of intensive investigation. We explain some of the current issues relating to the study of the maximal subgroups of classical groups.

## 1. INTRODUCTION

The ATLAS of finite simple groups has paved a path of accessibility to the theory and structure of the finite simple groups for non-group theorists. At the same time it continues to provide specialists with detailed and delicate information which is needed in current topics of interest. The study of the maximal subgroups of finite groups provides a very good illustration of this and the reader is referred to Aschbacher's article in these proceedings [1] for a general overview and context. The survey article by Tiep [36] also provides an excellent introduction to this topic as well as a host of applications. In this note we remark on the current state of affairs of the classification of the maximal subgroups of the finite classical groups and illustrate things with some examples.

We recall that a finite group  $G$  is **almost simple** if it contains a unique minimal normal subgroup  $S$  which is nonabelian and simple. A perfect group  $G$  (i.e.  $G = [G, G]$ ) is **quasisimple** if  $G/Z(G)$  is nonabelian simple. The classification of the finite simple groups implies that the Schreier conjecture holds; that is  $\text{Aut}(S)/S$  is solvable for all finite nonabelian simple groups  $S$ . By  $G^\infty$  we denote that last term of the derived series of  $G$ . So if  $G$  is almost simple, then  $S = G^\infty$  is nonabelian and simple.

We begin by recalling the main theorem of [4].

**Theorem 1.1** (Aschbacher - Scott 1985). *For the solution of the maximal subgroup problem for general finite groups it suffices to*

- (1) *determine the conjugacy classes of maximal subgroups of the almost simple groups; and*
- (2) *determine  $H^1(G, V)$  for all quasisimple groups  $G$  and all irreducible  $\mathbb{F}_\ell G$ -modules  $V$ , where  $\ell$  is a prime divisor of  $|G|$ .*

To see the relevance of the second problem we recall that  $|H^1(G, V)|$  is equal to the number of equivalence classes of complements to  $V$  in  $V \rtimes G$ , and that when  $G$  acts irreducibly on  $V$  then  $G$  is a maximal subgroup of  $V \rtimes G$ . We refer the reader to [14] for recent progress in this area.

We recall that the classification of the finite simple groups implies that finite nonabelian simple groups are either alternating, sporadic or of Lie type. The Lie type groups are further subdivided into exceptional and classical types. Aschbacher's article [1] describes the current state of affairs for the alternating and the exceptional Lie type groups.

The maximal subgroups of all but the largest sporadic group were classified between 1965 and 1998. The ATLAS [5] and Wilson's book [39] contain all the references to the

original articles. The maximal subgroups of the Monster  $\mathbb{M}$  have not yet been completely classified. The maximal  $p$ -local subgroups for odd primes  $p$  were classified in [37]. In addition, Table 5.6 in [39] features a list of known maximal subgroups of the Monster. Also recorded there is the fact that any maximal subgroup  $H$  not listed in Table 5.6 must necessarily be almost simple with  $H^\infty$  isomorphic to one of  $\mathrm{PSL}_2(13)$ ,  $\mathrm{PSU}_3(4)$ ,  $\mathrm{PSU}_3(8)$ ,  $\mathrm{Sz}(8)$ ,  $\mathrm{PSL}_2(8)$ ,  $\mathrm{PSL}_2(16)$ ,  $\mathrm{PSL}_2(27)$ . In his most recent article on the subject Wilson [38] deals with the cases  $\mathrm{Sz}(8)$  and  $\mathrm{PSL}_2(27)$ .

The starting point for the description of the maximal subgroups of the finite classical groups is Aschbacher's theorem [2]. To state the theorem we let  $k$  be a field of characteristic  $\ell$  and let  $X$  be a classical group with natural module  $V = k^m$ . Using  $V$ , Aschbacher defines eight families  $\mathcal{C}_i(X)$  of “geometric” subgroups of  $X$ , some of which we describe in more detail below.

**Theorem 1.2** (Aschbacher 1984). *If  $H \leq X$  is maximal, then either  $H \in \mathcal{C}_i(X)$ , or  $H \in \mathcal{S}(X)$  meaning that*

- (1)  $H^\infty$  is quasisimple;
- (2)  $H^\infty$  acts absolutely irreducibly on  $V$ ;
- (3) the action of  $H^\infty$  on  $V$  can not be defined over a smaller field;
- (4) any bilinear, quadratic or sesquilinear form on  $V$  that is stabilized by  $H^\infty$  is also stabilized by  $X$ .

Aschbacher's theorem does not imply that a maximal member of some family  $\mathcal{C}_i(X)$  or  $\mathcal{S}(X)$  is maximal, nor does it classify the maximal members of the families. In their book Kleidman and Liebeck [20] determine the conjugacy classes of maximal members of  $\mathcal{C}_i(X)$  and for  $\dim(V) \geq 13$  determine when a maximal member from a class  $\mathcal{C}_i(X)$  is in fact maximal in  $X$ . For  $\dim(V) \leq 12$  the recent book by Bray, Holt and Roney-Dougal [6] explicitly determines the maximal subgroups of  $X$ .

For  $\dim(V) \geq 13$ , this leaves the question of when a member of class  $\mathcal{S}(X)$  is in fact maximal in  $X$ ?

## 2. MAXIMALITY OF MEMBERS OF $\mathcal{S}(X)$ : AN OVERVIEW

We now consider a finite classical group  $X$  and  $H \in \mathcal{S}(X)$ . So  $H^\infty$  is quasisimple acting absolutely irreducibly on  $V$ . Replacing  $H$  with  $N_X(H)$  if necessary, we may assume without loss that  $H = N_X(H)$ . We must consider possible obstructions  $H < G < X$  to the maximality of  $H$  in  $X$ . In Table 1 we summarize the information provided in this section. The color coding in the table is designed to give the reader a feel for how abundant obstructions of a given type are. The colors are black, blue and red going from cold to hot, with hot indicating an abundance of obstructions, blue indicating few and black indicating no obstructions. We now explain the row and column labels.

The classification of finite simple groups implies that the possible choices for  $H^\infty$  are **S**poradic, **A**lternating, **C**lassical or of **E**Xceptional Lie type. If  $H$  is a classical or an exceptional group of Lie type we further distinguish whether or not the defining characteristic  $p$  of  $H$  is equal to  $\ell$ , the defining characteristic of  $X$ . In Table 1 the columns are labeled by the possible 6 choices for the type of  $H$  and are **S**por, **A**lt, **CL d**, **CL c**, **EX d**, **EX c**, in order of appearance. So for example **CL d** means classical group in defining characteristic, that is  $p = \ell$ , and **EX c** means exceptional groups of Lie type in cross characteristic, that is  $p \neq \ell$ . We have chosen to place the column labels in the bottom row so as to emphasize the fact that  $H$  is a subgroup of  $G$ . The rows of Table 1 are labeled by the possible types of  $G$ .

The definitions of the families  $\mathcal{C}_i(X)$  in Aschbacher's theorem 1.2 are such that  $H < G$  and  $H \in \mathcal{S}(X)$  implies that  $G$  is not a member of  $\mathcal{C}_1(X) \cup \mathcal{C}_3(X) \cup \mathcal{C}_5(X) \cup \mathcal{C}_8(X)$ .

Thus the possible obstructions  $G$  to the maximality of  $H$  in  $X$  must lie in

$$\mathcal{C}_2(X) \cup \mathcal{C}_4(X) \cup \mathcal{C}_6(X) \cup \mathcal{C}_7(X) \cup \mathcal{S}(X),$$

giving a total of 10 possibilities for  $G$ .

We have arranged the table in such a way that the possible obstructions  $G$  of  $H$  lie in the column over the type of  $H$ . We consider the possible  $G$  lying over  $H$  proceeding row wise.

TABLE 1. Obstructions  $G$  to the maximality of  $H \in \mathcal{S}(X)$

$\mathcal{C}_2$	HHM	DM/NN	Seitz	HHM	Seitz	HHM	induced
$\mathcal{C}_4$	MT	BK	Stein	MT	Stein	MT	tensor prod
$\mathcal{C}_6$	MT	and	Bray	inde	pend	ently	$r^{1+2n}\text{Sp}_{2n}(r)$
$\mathcal{C}_7$	$\leftarrow$	—	MT	—	—	$\rightarrow$	tensor ind
<b>Spor</b>		Hu	LSS		LSS		$\uparrow$
<b>Alt</b>	S/KW BK/KS KTS	S/KW/JS BK/KS KTS	S/KW BK/KS KTS	S/KW BK/KS KTS	S/KW BK/KS KTS	S/KW BK/KS KTS	$\downarrow$
<b>CL d</b>		Hu	D/S/T	MRT	D/S/T	MRT	branching
<b>CL c</b>		Hu	LSS	Seitz	LSS	Se/S/N	rules
<b>EX d</b>		Hu	D/S/T	MRT	D/S/T	MRT	$\downarrow$
<b>EX c</b>		Hu	LSS	Seitz	LSS	Seitz	$\downarrow$
$G / H$	<b>Spor</b>	<b>Alt</b>	<b>CL d</b>	<b>CL c</b>	<b>EX d</b>	<b>EX c</b>	$H$ rep'n

HHM = Hiss Husen Magaard, MT = Magaard Tiep, MRT = Magaard Röhrle Testerman,

BK = Bessenrodt Kleshchev, DM/NN = Djoković Malzan / Nett Noeske, D/S/T =

Dynkin/Seitz/Testerman, Hu = Husen, JS = Jantzen Seitz, S/KW = Saxl/Kleidman Wales,

BK/KS/KST = Brundan Kleshchev/Kleshchev Sheth/Kleshchev Sin Tiep, LSS = Liebeck

Saxl Seitz, Se/S/N = Seitz/ Schaeffer-Fry / H.N. Nguyen

**2.1.  $\mathcal{C}_2$  type obstructions.** If  $G$  is a  $\mathcal{C}_2(X)$  obstruction to the maximality of  $H$ , then  $H^\infty$  must stabilize a direct sum decomposition of  $V$  into isometric subspaces. This implies that the corresponding  $\bar{k}H^\infty$  character, where  $\bar{k}$  denotes the algebraic closure of  $k$ , must necessarily be imprimitive. Thus the space  $\bar{V} := V \otimes_k \bar{k}$  possesses an  $H$ -invariant decomposition  $\sum \bar{V}_i$  with block stabilizer  $H_1$ .

All imprimitive irreducible representations of alternating and symmetric groups of degree  $n$  with  $\ell > n$  were classified by Djoković and Malzan in [11] and [12]. For  $\ell \leq n$  Nett and Noeske [31] generalized the results of Djoković and Malzan to Schur covers of  $A_n$  and  $S_n$  and characteristics  $\ell \leq n$ . If  $\ell > n$  there are exactly three generic families of imprimitive irreducible characters. These examples persist when  $\ell \leq n$ , however there may be additional examples when  $H$  is a Schur cover and the block stabilizer is the inverse image of an intransitive maximal subgroup of  $H/Z(H)$ . Whether or not there are additional examples in case  $H$  is a Schur cover of  $A_n$  or  $S_n$  and  $\ell \leq n$  is an open question.

Seitz [33] shows that there are no imprimitive irreducible representations of groups of Lie type in defining characteristic apart from the Steinberg representation for  $\text{SL}_2(5)$ ,  $\text{SL}_2(7)$ ,  $\text{SL}_3(2)$  and  $\text{Sp}_4(3)$ . It turns out that none of these representations lead to  $\mathcal{C}_2$  obstructions because the field of definition needed for the obstruction is bigger than the minimal field of definition for the module  $V$ . For example the Steinberg module of  $\text{SL}_3(2)$

leads to the embedding  $\mathrm{SL}_3(2) \in \mathcal{S}(\Omega_8^-(2))$ . As can be seen in the ATLAS [5] the group  $\mathrm{SL}_3(2)$  is maximal in  $\Omega_8^-(2)$ . The Steinberg module  $V$  is induced from a non-principal linear character of the Forbenius group of order 21 and hence the imprimitivity decomposition of  $V$  can only exist over fields containing a 3'rd root of unity, which  $\mathbb{F}_2$  clearly does not.

All imprimitive irreducible modules for sporadic simple groups are classified in Chapter 3 of Hiss, Husen, Magaard [15]. With one exception Chapter 5 classifies all imprimitive irreducible modules for  $H$  a quasisimple group of Lie type which possess an exceptional Schur multiplier. The open case here is  $H \cong 2^2E_6(2)$  and  $\ell$  is a divisor of  $|H|$ .

Hiss, Husen, Magaard [15] further show that if  $H$  is quasisimple of Lie type and  $H$  does not possess an exceptional Schur multiplier, then any imprimitive irreducible cross-characteristic  $H$ -module is Harish–Chandra induced; i.e.  $H_1$  is a parabolic subgroup of  $H$  such that the unipotent radical of  $H_1$  acts trivially on  $V_1$ . It is then shown that as the Lie rank of  $H$  tends to infinity the proportion of imprimitive representations tends to 1. For  $\ell = 0$  and  $H$  the fixed points of an algebraic group with connected center all imprimitive irreducible  $\bar{k}H$ -modules are described in [15] in terms of Harish–Chandra and Deligne–Lusztig series.

At this point some words of caution are in order. While the imprimitivity condition for the  $H$ -module  $\bar{V}$  is necessary for the existence of a potential  $\mathcal{C}_2(X)$ -type obstruction, it may not be sufficient. Firstly it may happen that an imprimitivity decomposition of  $\bar{V}$  may not be definable over the field  $k$ , secondly the imprimitivity decomposition may be incompatible with the form on  $V$  defining  $X$ , and thirdly the class  $M^H$  where  $M$  is the block stabilizer of the imprimitivity decomposition of  $H$  may not be invariant under the action of  $N_X(H)$ .

Consider for example the group  $H := \mathrm{SL}_2(q)$  with  $q$  odd and  $(q, \ell) = 1$ . If  $q \geq 13$ , then the smallest index subgroups of  $H$  are the Borel subgroups of index  $q + 1$  in  $H$ . As  $\chi(1) \leq q + 1$  for all  $\chi \in \mathrm{Irr}(H)$  we deduce that imprimitive irreducible  $H$  characters must all have degree  $q + 1$ .

Let  $B = UT$  be a Borel subgroup of  $H$  with unipotent radical  $U$  and split torus  $T$ . The ordinary characters of degree  $q + 1$  are all Harish–Chandra induced from  $B$ . They are in fact Deligne–Lusztig characters of the form  $R_{T, \Theta}$  where  $\Theta$  is a character of  $T$  of order  $> 2$ . If  $\bar{V}$  is a module affording the reduction of  $R_{T, \Theta} \bmod \ell$ , then the field of definition is determined by the action of  $T$  in its action on  $C_{\bar{V}}(U)$ , as  $T$  permutes the nontrivial characters of  $U$  in two orbits of size  $(q - 1)/2$ . The action of  $T$  on  $C_{\bar{V}}(U)$  extends to  $N_H(T)$ . The latter group acts as a dihedral group  $D$  whose order is determined by the order  $d$  of  $\Theta(t)$  where  $t$  is a generator for  $T$ . The smallest field over which the imprimitivity decomposition of  $\bar{V}$  can be defined must contain a  $d$ 'th root of unity  $\zeta_d$ , whereas the field of definition of  $R_{T, \Theta}$  is the smallest field containing  $\zeta_d + \zeta_d^{-1}$ . Whether or not these fields are identical depends on  $d$  and  $\ell$ .

Now suppose that  $d$  and  $\ell$  are such that the smallest extension  $k$  of  $\mathbb{F}_\ell$  containing  $\zeta_d + \zeta_d^{-1}$  contains  $\zeta_d$ . Then  $R_{T, \Theta}$  is defined over  $k$  as is the imprimitivity decomposition and thus  $H < \mathbb{Z}_d \wr S_{q+1} < \mathrm{GL}_{q+1}(k)$ . Moreover the action of  $H$  on the summands of the imprimitivity decomposition is doubly transitive. On the other hand we know that  $R_{T, \Theta}$  is a self dual character. Depending on whether or not  $Z(H) \leq \ker(\Theta)$  the Frobenius Schur indicator is  $+1$  respectively  $-1$ . If the indicator is  $-1$ , then  $H \in \mathcal{S}(\mathrm{Sp}_{q+1}(k))$ . On the other hand the largest possible stabilizer in  $\mathrm{Sp}_{q+1}(k)$  of a decomposition into one spaces is contained in the group  $(k^* \wr S_2) \wr S_{(q+1)/2}$ . However  $H$  does not possess a permutation representation of degree less than  $q + 1$  and is quasisimple and therefore can not embed

into  $(k^* \wr S_2) \wr S_{(q+1)/2}$ . So  $H \in \mathcal{S}(\mathrm{Sp}_{q+1}(k))$  while acting imprimitively on  $V = k^{q+1}$  does not possess a  $\mathcal{C}_2(\mathrm{Sp}_{q+1}(k))$  obstruction. We also see that a similar statement holds when the indicator is  $+1$ . Thus we have obtained the following.

**Lemma 2.1.** *Let  $H = \mathrm{SL}_2(q)$ ,  $q > 13$ ,  $(q, \ell) = 1$  and let  $V$  be an irreducible  $\mathbb{F}_\ell H$ -module of dimension  $q + 1$ . Let  $X = \Omega(V)$  if the Frobenius Schur indicator of  $V$  is  $+$  and  $X = \mathrm{Sp}(V)$  otherwise. The  $\mathbb{F}_\ell H$ -module  $V$  is imprimitive and  $N_X(H) \in \mathcal{S}(X)$  is not  $\mathcal{C}_2(X)$ -obstructed in  $X$ .*

We note without proof that under the hypothesis of the lemma  $N_X(H)$  is maximal in  $X$ . In striking contrast to the previous lemma is perhaps the following

**Fact:** Let  $H = \mathrm{SL}_3(q)$ ,  $q > 3$ ,  $(q, \ell) = 1$  and let  $V$  be an irreducible  $\mathbb{F}_\ell H$ -module of dimension  $q^2 + q + 1$  and let  $X$  be a classical group with natural module  $V$  such that  $H \in \mathcal{S}(X)$ . The  $H$ -module  $V$  is imprimitive and in fact  $V = \mathrm{Ind}_P^G(\Theta)$  where  $P$  is a maximal parabolic subgroup of  $H$  of index  $q^2 + q + 1$  and  $\Theta$  is a linear character of  $P$ . The opposite parabolic  $P^{op}$  of  $P$  is conjugate to  $P$  in  $\mathrm{Aut}(H)$  but is not conjugate to  $P$  in  $H$ . So we see that  $V = \mathrm{Ind}_{P^{op}}^G(\Theta)$  which implies that  $H$  is contained in two distinct members of  $\mathcal{C}_2(X)$ .

**Warning:** We already saw that the fact that  $H$ -module  $V$  is imprimitive does not automatically imply that  $H \in \mathcal{C}_2(X)$ . Even worse it may happen that  $H \in \mathcal{C}_2(X)$  but  $N_X(H) \notin \mathcal{C}_2(X)$ . If  $H = \mathrm{SL}_3(q)$  and  $V$  are as in our example above and  $N_X(H)$  contains an automorphism interchanging the two  $H$ -classes of maximal parabolic subgroups, then  $N_X(H)$  is not  $\mathcal{C}_2(X)$ -obstructed. We will not prove this here but will illustrate this phenomenon with the case  $H = \mathrm{SL}_3(5) = \mathrm{PSL}_3(5)$  and  $\ell \notin \{2, 3, 5, 31\}$ .

The ATLAS-characters  $\chi_3, \chi_4, \chi_5$  of  $H$  all have degree 31 and indicators  $+1, 0, 0$  respectively. The character values  $\chi_4$  and  $\chi_5$  involve a primitive fourth root of unity. Now a representation affording  $\chi_3$  embeds  $H$  into  $\mathrm{O}_{31}(\overline{\mathbb{F}}_\ell)$ , where  $\overline{\mathbb{F}}_\ell$  denotes the algebraic closure of  $\mathbb{F}_\ell$ . Also the representation is defined over  $\mathbb{F}_\ell$  and  $H$  is simple, so in fact we see that  $H \leq \Omega_{31}(\ell)$  and consequently  $H \in \mathcal{S}(X)$  for all  $X$  with  $X^\infty \cong \Omega_{31}(\ell)$ . Arguing similarly we see that representations affording  $\chi_4$  or  $\chi_5$  embed  $H \in \mathcal{S}(X)$ , where  $X^\infty = \mathrm{SL}_{31}(\ell)$  respectively  $\mathrm{SU}_{31}(\ell)$  when  $\ell \equiv 1 \pmod{4}$  respectively  $\ell \equiv 3 \pmod{4}$ . In every case the embedding of  $H$  into  $X$  lies in two  $\mathcal{C}_2(X)$ -subgroups of  $X^\infty$ . More specifically we have:

$$H = \mathrm{PSL}_3(5) \leq \mathbb{Z}_2^{30} \rtimes A_{31} \leq \Omega_{31}(\ell),$$

$$H \leq \mathbb{Z}_{\ell-1}^{30} \rtimes S_{31} \leq \mathrm{SL}_{31}(\ell), \quad \text{if } \ell \equiv 1 \pmod{4}$$

and

$$H \leq \mathbb{Z}_{\ell+1}^{30} \rtimes S_{31} \leq \mathrm{SU}_{31}(\ell), \quad \text{if } \ell \equiv 3 \pmod{4}.$$

The graph automorphism of  $H$  permutes the characters  $\chi_4$  and  $\chi_5$  and thus if  $X = \mathrm{SL}_{31}(\ell)$ , respectively  $X = \mathrm{SU}_{31}(\ell)$ , then no element of  $N_X(H)$  induces a graph automorphism on  $H$  and thus  $N_X(H) \leq \mathbb{Z}_{\ell \pm 1}^{30} \rtimes S_{31} \leq X$  is not maximal by reason of being  $\mathcal{C}_2(X)$ -obstructed.

On the other hand when  $H \leq \Omega_{31}(\ell)$ , then the representation of  $H$  on  $V$  extends to the graph automorphism  $\sigma$  of  $H$  in two ways, one of which is such that  $\langle H, \sigma \rangle \leq \mathrm{SO}_{31}(\ell)$ .

Thus we obtain:

**Lemma 2.2.** *Let  $H = \mathrm{PSL}_3(5)$ .*

- (1) If  $X = \mathrm{SL}_{31}(\ell)$  or  $\mathrm{SU}_{31}(\ell)$ , then  $N_X(H) \in \mathcal{S}(X)$  is contained in a member of  $\mathcal{C}_2(X)$  and hence is not maximal in  $X$ .
- (2) If  $X = \mathrm{SO}_{31}(\ell)$ , then  $H \in \mathcal{S}(X)$  and  $H$  is contained in two  $\mathcal{C}_2(X)$  subgroups which are permuted by  $N_X(H)$ . The subgroup  $N_X(H)$  is maximal in  $X$ .

We conclude this subsection by observing that the complete enumeration of  $\mathcal{C}_2$ -type obstructions for cross characteristic representations of  $H$  will require the resolution of the following issues:

- (1) A complete classification of imprimitive irreducible representations when  $\ell = 0$ . The key question here is what happens when we drop the hypothesis that  $H$  is the fixed points of an algebraic group with connected center? This is an ongoing project with Gerhard Hiss.
- (2) A complete classification of imprimitive irreducible representations when  $\ell$  divides  $|H|$ .
- (3) For each irreducible imprimitive representation of  $H$  determine whether or not the minimal fields of definition of  $H$  and the imprimitivity decomposition coincide.
- (4) For each irreducible imprimitive representation of  $H$  determine whether or not the imprimitivity decomposition is compatible with the  $X$ -invariant form.
- (5) Determine whether or not  $N_X(H)$  fixes  $M^H$ , where  $M$  is the block stabilizer of the imprimitivity decomposition of  $H$  on  $V$ .

**2.2.  $\mathcal{C}_4$  and  $\mathcal{C}_7$  type obstructions.** If  $G$  is a  $\mathcal{C}_4(X)$  respectively a  $\mathcal{C}_7(X)$  obstruction to the maximality of  $H$ , then  $H^\infty$  must stabilize an asymmetric respectively a symmetric tensor product decomposition of  $\bar{V}$ ; i.e., a tensor product of spaces of unequal, respectively equal, dimensions. This implies that the corresponding  $kH^\infty$  character, must necessarily be a Kronecker product of characters. We remind the reader that, as in the previous section, the existence of a Kronecker product factorization of a character for  $H^\infty$  is not sufficient to guarantee the existence of a  $\mathcal{C}_4$  type obstruction. The issues are much the same as outlined in the previous section, fields of definition, invariant forms and the action of the automorphisms induced by elements of  $H_X(H^\infty)$  on  $H^\infty$ -modules. We now comment on the current state of knowledge.

Factorizations of irreducible characters of alternating and symmetric groups and their Schur covers into Kronecker products of irreducible characters are very rare. The relevant references are Kleshchev and Bessenrodt [8], [9] and [10] as well as Kleshchev and Tiep [25]. It is an open question as to which of these factorizations do in fact lead to  $\mathcal{C}_4$ -obstructions.

Magaard and Tiep [28] observe, by checking the tables available in GAP [13] that 18 of the 26 sporadic groups possess irreducible characters which factor as a Kronecker product. They also show that if  $H$  is of Lie type of characteristic  $p$  and  $p \neq \ell$ , then factorizations of irreducible characters into Kronecker products can exist only if  $H$  is defined over a field of size at most 5 or  $\mathrm{Sp}_{2r}(q)$  with  $q$ -even.

It is also shown that certain tensor products of Weil representations of  $\mathrm{Sp}_{2r}(3)$  are irreducible. We remind the reader that the Weil representations of  $H = \mathrm{Sp}_{2r}(3)$  are those of dimensions  $(3^r \pm 1)/2$ . These restrict irreducibly modulo every prime and are not self dual. In [28] they are labeled  $\xi, \xi^*, \eta$ , and  $\eta^*$  and it is shown that the characters  $\xi\eta^*$  and  $\xi^*\eta$  are dual to each other and irreducible. An analysis similar to the one in the previous section shows that if  $X = \mathrm{SL}_{\xi(1)\eta(1)}(\ell)$ , then  $H \in \mathcal{S}(X)$  for suitable primes  $\ell$ , and moreover  $N_X(H)$  is  $\mathcal{C}_4(X)$  obstructed.



The only other known infinite families of irreducible tensor products of cross characteristic representations are certain Weil representations of  $\mathrm{Sp}_{2r}(5)$  and  $\mathrm{SU}_{r+1}(2)$ . Which of these factorizations lead to actual  $\mathcal{C}_4$  obstructions is still an open question. We also mention that the expectation is that no other infinite families of irreducible cross characteristic tensor decomposable representations exist.

The situation is very different when  $H$  is of Lie type of characteristic  $p = \ell$ . Here Steinberg's tensor product theorem implies that only  $p$ -restricted representations of  $H$  may not possess factorizations into tensor products. In addition, the groups  $\mathrm{Sp}_{2r}(2^f)$  and  $F_4(2^f)$  and  $G_2(3^f)$  also possess  $p$ -restricted representations which factor properly into tensor products. Thus if  $H(q)$  is a group of fixed Lie type defined over  $\mathbb{F}_q$ ,  $q = p^f$  of untwisted Lie rank  $r$ , then the total number of tensor indecomposable irreducible defining characteristic characters is bounded above by  $f p^r$  while the total number of defining characteristic characters is  $q^r$  and thus  $\lim_{f \rightarrow \infty} \frac{f p^r}{q^r} = 0$ ; justifying the color red in Table 1.

Here, as in the  $\mathcal{C}_2$  case, it may happen that factorizations of an irreducible character into a Kronecker product are incompatible with the form defining  $X$ . It may also happen that the fields of definition of the factors are larger than the field of definition of the character being factored. These are highly nontrivial issues and large sections of chapter 5 of [6] are devoted to these issues. For more detail we refer the reader to Proposition 5.1.14 as well as Propositions 5.4.20 and 5.4.21.

We close this subsection with a sporadic group example which exhibits the problems that can arise from the field of definition and the invariant form.

Let  $H \cong M_{24}$ . The ordinary character  $\chi_{26}$  factors as

$$\chi_3 \otimes \chi_5 = \chi_3 \otimes \chi_6 = \chi_{26} = \chi_4 \otimes \chi_5 = \chi_4 \otimes \chi_6.$$

However the Frobenius–Schur indicator of  $\chi_{26}$  is  $+1$  and all character values are rational, whereas the characters  $\chi_3, \chi_4, \chi_5$ , and  $\chi_6$  all possess irrationalities and have indicator 0. So we see that  $M_{24} \in \mathcal{S}(\Omega_{10395}(\ell))$ , and any module affording  $\chi_{26}$  factors as a tensor product over an algebraic closure of the prime field. The factorization into a tensor product is defined over the prime field if and only if the ATLAS irrationalities  $b7$  and  $b15$  lie in the prime field. Even if  $b7$  and  $b15$  lie in the prime field, which implies that the reduction of  $\chi_3, \chi_4, \chi_5$ , and  $\chi_6$  modulo  $\ell$  are defined over the prime field, then the fact that the indicators of  $\chi_3, \chi_4, \chi_5, \chi_6$  are zero implies that for  $\ell$  coprime to  $|M_{24}|$  the embedding  $M_{24} \in \mathcal{S}(\Omega_{10395}(\ell))$  is never  $\mathcal{C}_4$  obstructed.

**2.3.  $\mathcal{C}_6$  type obstructions.** The  $\mathcal{C}_6$  type obstructions to  $H$  in  $X$  are normalizers of extraspecial  $r$  groups and certain symplectic type 2-groups, where  $r \neq \ell$  is a prime. The precise setup that we use is described in Kleidman and Liebeck [20] and is as follows.

Let  $r$  be a prime, and  $E(r)$  be an extraspecial group of order  $r^{1+2s}$  and exponent  $r(2, r)$ . When  $r = 2$ , then for each choice of  $s$  there are exactly two nonisomorphic such groups which we distinguish by a subscript  $\epsilon = \pm$ . Also we define

$$E(4) := \mathbb{Z}_4 \circ E(2)_+ \cong \mathbb{Z}_4 \circ E(2)_-.$$

For  $E(t)$  with  $t > 2$  set  $e$  be the smallest integer such that  $t \equiv 1 \pmod{\ell^e}$ . If  $e$  is even, then set  $X^\infty = \mathrm{SU}_{r^s}(\ell^e)$  and if  $e$  is odd set  $X^\infty = \mathrm{SL}_{r^s}(\ell^e)$ . For  $E(2)_-$  set  $X^\infty = \Omega_{2s}^-(\ell)$  if  $s > 1$  and  $\mathrm{SL}_{2s}(\ell)$  otherwise, and for  $E(2)_+$  set  $X^\infty = \Omega_{2s}^+(\ell)$ .

The representation theory of  $E(t)$  implies that  $E(t)$  acts absolutely irreducibly and faithfully on the natural module  $V$  of  $X$  and moreover, the representation of  $E(t)$  is not definable over any subfield of  $\mathbb{F}_{\ell^e}$ . The group  $G := N_X(E(t))$  is a maximal element of



$\mathcal{C}_6(X)$ . Also

$$N_{X^\infty}(E(t))/(C_{X^\infty}(E(t))E(t)) \cong \mathrm{Sp}_{2s}(\frac{t}{(2,t)})$$

if  $t > 2$ , whereas

$$[N_{X^\infty}(E(2)_-)/(C_{X^\infty}(E(2)_-)E(2)_-)]^\infty \cong \Omega_{2s}^-(2),$$

and

$$[N_{X^\infty}(E(2)_+)/(C_{X^\infty}(E(2)_+)E(2)_+)]^\infty \cong \Omega_{2s}^+(2).$$

Suppose now that  $G$  is a  $\mathcal{C}_6$  obstruction of  $H$ , then Magaard and Tiep [29] observe that the following must be true:

- (1)  $H^\infty$  has an absolutely irreducible representation of degree  $r^s$  in characteristic  $\ell \neq r$ .
- (2)  $H^\infty$  has an absolutely irreducible representation of degree  $r^s$  in characteristic 0.
- (3)  $H^\infty$  has a representation of degree  $2s$  in characteristic  $r$  such that the center of the representation lies in  $Z(E)$ .

These conditions are so restrictive that the only potential possibilities for  $F^*(H)$ , are  $2A_{4m}$ , ( $m \geq 2$ ),  $2M_{12}$ ,  $2A_5$ ,  $2A_6$ ,  $\mathrm{SL}_2(17)$  and  $2\mathrm{Sp}_6(2)$ . A more detailed analysis reveals that all but the first two cases are impossible. In the first two cases  $r = 2$  and  $V$  is a basic spin module, respectively a module of dimension 32 for  $H$ . In fact the natural embedding of  $2M_{12}$  in  $2A_{12}$  restricts irreducibly on the basic spin module of  $2A_{12}$ .

These results have also been independently obtained in unpublished work of John Bray. The complete proof of the results above will appear in a forthcoming paper of Bray, Magaard and Tiep.

**2.4.  $\mathcal{S}$  type obstructions.** The obstructions  $G$  of type  $\mathcal{S}$  of  $H$  can only arise when  $G^\infty$  possesses an absolutely irreducible  $\mathbb{F}_\ell$  module  $V$  whose restriction to  $H^\infty$  is absolutely irreducible, which is a branching problem. As a first step towards a classification of  $\mathcal{S}$ -type obstructions to the maximality of  $H \in \mathcal{S}(X)$  we must first find all triples  $(H, G, V)$  where  $H$  and  $G$  are quasisimple, and  $V$  is an absolutely irreducible  $G$ -module whose restriction to  $H$  is also absolutely irreducible.

If  $G$  is sporadic,  $H < G$ ,  $H, G \in \mathcal{S}(X)$ , then there exists  $\varphi \in \mathrm{IBr}_\ell(G)$  which restricts irreducibly to  $H$ . If  $\ell = 0$ , the character table of  $G$  and those of its maximal subgroups are known and available in GAP [13]. So all potential triples  $(H, G, V)$  can be extracted fairly easily. To a large extent the same is true when  $\ell$  is a divisor of  $|G|$ . However the computation of the modular character tables of the large sporadic groups and primes for which the Sylow subgroups are not cyclic is challenging and an active area of research. As indicated in Table 1 certain branching problems which are relevant for the determination of the maximal subgroups of the classical groups have been already been addressed by Husen [16] and Liebeck, Saxl, Seitz [26].

It is worth remarking that Husen [16], [17], [18] fixed the group  $H^\infty \cong 2.A_n$  or  $A_n$  and allowed  $G$  to vary, meaning that his analysis treated a column of Table 1. More commonly analyses of branching rules fix  $G$  and allow  $H$  to vary; i.e. proceeding along a row of Table 1.

Branching problems for alternating and symmetric groups and their Schur covers have a venerable history going back to Young. Saxl [33] and Kleidman, Wales [21] completely solved the characteristic zero case for alternating and symmetric groups respectively their Schur covers. For  $\ell > 3$  and  $G = S_n$  Brundan and Kleshchev [7], and for  $G = A_n$  Kleshchev and Sheth [22], [23], solve the branching problem completely. Recent work of Kleshchev, Sin, Tiep [24] goes a long way towards solving the branching problems when

$\ell = 2$  or  $3$  and  $G = A_n$  or  $S_n$ . Kleshchev and Tiep [25] nearly obtained a complete solution in the case  $G = 2.A_n$  or  $2.S_n$  and  $\ell < n$ .

The largest number of cases to consider arise when both  $H$  and  $G$  are of Lie type. The optimal subdivision of cases here is determined by the characteristics of  $H$ ,  $G$  and  $X$ . Let  $p$  and  $s$  denote the characteristics of  $H$  and  $G$  respectively and recall that the characteristic of  $X$  is  $\ell$ .

If  $p \neq s \neq \ell$ , but  $p = \ell$ , then the complete classification of examples can be found in Liebeck, Saxl, Seitz [26]. We note that there are very few examples in this case. The expectation is that there are still fewer examples in case  $p \neq \ell$ . However this easy case has not yet been considered.

If  $p = s \neq \ell$ , then Seitz [34] showed that there are exactly four types of possibilities for  $(H, G)$ . The possibilities  $(G_2(2^a), \text{Sp}_6(2^a))$  were completely classified by Schaeffer-Fry [32]. Also see Tiep [36] for further discussion of this case.

If  $p = s = \ell$ , then this is a classical result of Dynkin when  $\ell = 0$ . Seitz [33] generalized Dynkin's work to  $\ell \neq 0$  for  $G$  classical, and the case  $G$  exceptional was treated by Testerman [35]. Again these types of examples are relatively rare.

The final case is the case  $p \neq s = \ell$ . The first reduction is due to Magaard, Röhrle, Testerman [30] and asserts the following.

**Theorem 2.3.** *Suppose  $H = Y_r(q)$ ,  $q = p^f$ , is a group of Lie type  $Y$ , untwisted rank  $r$ , and characteristic  $p$ ,  $W = k^m$  where  $k$  is an algebraically closed field of characteristic  $\ell$  with  $(\ell, p) = 1$  and  $\rho : H \rightarrow \text{GL}(W)$  an irreducible representation of  $H$ . If the Frobenius Schur indicator of  $\rho$  is zero set  $G = \text{SL}(W)$ , if it is  $-1$  set  $G = \text{Sp}(W)$ , and if it is  $+1$  set  $G = \text{Spin}(W)$ . Let  $V$  be a finite-dimensional irreducible  $kG$ -module.*

*If  $W$  is  $Q$ -linear large, then  $V|_{H\rho}$  is reducible unless either*

- *$V$  is restricted and equivalent to  $W$  or  $W^*$ , or*
- *$q \leq 3$ ,  $H$  is not a central extension of  $\text{PSL}_n(q)$ , and  $V$  is a Frobenius twist of  $A \otimes B^\delta$ , where  $B^\delta$  is a Frobenius twist of  $B$  and  $A, B \in \{W, W^*\}$  such that  $A|_H$  and  $B^\delta|_H$  are inequivalent  $kH$ -modules.*

We remark that the  $Q$  linear large hypothesis on the  $H$ -module  $W$  implies that the degree of  $\rho$  is not too much larger than the minimal degree of any nontrivial projective  $kH$ -module. We also remark that the  $\ell$ -restricted  $G$ -modules are all subquotients of  $W^{\otimes e_1} \otimes (W^*)^{\otimes e_2}$ . In ongoing work Magaard and Testerman can show that if  $H$  is linear or orthogonal then  $e_1 + e_2 \leq 4$ . For  $r > 4$  this will imply that in fact  $e_1 + e_2 = 2$  which in turn implies that the same conclusion as in Theorem 2.3 holds. More generally for  $H$  not linear or orthogonal preliminary results of Magaard and Testerman show that  $e_1 + e_2 \leq 3$  if  $q \geq 27$ . For  $q < 27$  they show that  $e_1 + e_2 \leq b(H)$  where  $b(H)$  is given explicitly and  $3 \leq b(H) \leq 19$ . The current aim is to improve the estimate to  $e_1 + e_2 \leq 2$ , whenever possible. In those cases where one can show that  $e_1 + e_2 \leq 2$  one can use the results of Magaard, Malle and Tiep [27] to achieve the conclusion of Theorem 2.3. We note that if  $W$  is a Weil module for  $H = \text{Sp}_{2r}(3)$ , then unpublished work of Tiep and Magaard (see also [36]) shows that  $\Lambda^3(W)$  and  $\text{Sym}^3(W)$  are irreducible  $H$ -modules, indicating that the bound  $e_1 + e_2 \leq 2$  can not hold universally.

We conclude this section with the general observation that when  $H^\infty$  is an alternating group or the Schur cover of an alternating group and  $H \in \mathcal{S}(X)$ , then the maximality of  $H$  is rarely obstructed. On the other hand if  $H$  is of Lie type and characteristic  $p$ , then the maximality of  $H$  is very often obstructed, either by a  $C_4(X)$  obstruction in case  $p = \ell$  or by a  $C_2(X)$  obstruction if  $p \neq \ell$ . Thus the contribution of the family  $\mathcal{S}(X)$

to the set of maximal subgroups of  $X$  may be much smaller than originally envisioned. However this means that the lattice of overgroups of minimal members of  $\mathcal{S}(X)$  may also be arbitrarily complicated, and in particular there may exist arbitrarily long chains of members of  $\mathcal{S}(X)$ . The irreducible embeddings of the smallest sporadic finite simple group into classical groups already hint at this.

### 3. IRREDUCIBLE REPRESENTATIONS OF $M_{11}$

In an ongoing project, Tung Le and the author are investigating the obstructions to the maximality of sporadic groups  $H$  in  $\mathcal{S}(X)$ . It is here where one expects the highest proportion, relative to the total number of equivalence classes of irreducible characters, of obstructions to maximality to exist.

The prime divisors of the order of  $H \cong M_{11}$  are 2, 3, 5, 11. For all other primes  $\ell$  the ordinary irreducible characters reduce irreducibly modulo  $\ell$ . We have recorded our results in Tables 6, 5, 4, 3 and 2 respectively. We use the labels from [5] and [19] to identify the irreducible representations of  $H$  and these are recorded in the first column of our tables.

For each character we consider an irreducible  $H$ -module  $V$  affording a character  $\varphi \in \text{IBr}_\ell(H)$ . A priori  $V$  is defined over an algebraically closed field of characteristic  $\ell$  and all we know is that  $H < \text{GL}(V)$ . Our first task is find the unique finite classical subgroup  $X < \text{GL}(V)$  for which  $H \in \mathcal{S}(X)$ . To this end we need to find the (modular) indicator and the minimal field of definition.

We may of course without loss assume that  $\ell \neq 0$  and hence the minimal field of definition is uniquely determined by the character values once a lifting of eigenvalues from characteristic  $\ell$  to characteristic zero has been fixed. Thus the minimal field of definition is the smallest degree extension of  $\mathbb{F}_\ell$  containing the images of the inverses of the lifts of  $\varphi(x)$  as  $x$  ranges over  $G$ . The irrational character values, in ATLAS notation, that we encounter are  $i2$  and  $b11$ . Thus our field of definition is either  $\mathbb{F}_\ell$  or  $\mathbb{F}_{\ell^2}$  depending on the irrationality and  $\ell$ . In the third column of Table 2 we have recorded the relevant congruences, sometimes using the Jacobi symbol.

If the Frobenius Schur indicator of the  $H$ -module  $V$  is  $+$  set  $X = \Omega^\epsilon(V)$ , if it is  $-$  set  $X = \text{Sp}(V)$ . If the Frobenius Schur indicator is zero set  $X = \text{SL}(V)$  or  $\text{SU}(V)$  depending on whether or not the dual of  $V$  is equal to  $V^\ell$ , the Galois twist of  $V$ ; i.e. the map induced by the field automorphism  $x \mapsto x^\ell$ . See the introductory section of the modular ATLAS [19] for a fuller explanation of this. Generally there is no character theoretic method to determine  $\epsilon$  in case  $X = \Omega^\epsilon(V)$  and  $\dim(V)$  is even. To determine the  $\epsilon$ 's that we require, we use [6] if  $\dim(V) \leq 12$ . We note that the permutation character of degree 55 of  $M_{11}$  in its action on the duads (sets of size 2) of 11 points. This character decomposes exactly like that of  $A_{11}$ ; i.e. the  $A_{11}$  permutation character of degree 55 decomposes as  $\chi_1 + \chi_2 + \chi_3$  and the restriction of each character to  $M_{11}$  stays irreducible. If  $M$  is a permutation module affording  $\chi_1 + \chi_2 + \chi_3$ , then the invariant form can be represented by the identity matrix  $I_{55}$  with respect to the natural permutation basis. The all ones vector  $m \in M$  is a basis for a submodule of  $M$  affording  $\chi_1$  and  $m^\perp$  is a module affording  $\chi_2 + \chi_3$ .

Restricting the form  $I_{55}$  means that the sign of  $m^\perp$  is  $\left(\frac{-55}{\ell}\right)$ . On the other hand the sign of the space  $m^\perp$  is  $\epsilon_{10}\epsilon_{44}$ . Combining this yields

$$\epsilon_{44} = \left(\frac{-55}{\ell}\right) \epsilon_{10} = \left(\frac{-55}{\ell}\right) \left(\frac{-11}{\ell}\right) = \left(\frac{5}{\ell}\right) \left(\frac{-11}{\ell}\right)^2 = \left(\frac{5}{\ell}\right).$$

$\chi$	Type of $X$	conditions	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
$\chi_2$	$\Omega_{10}^{\epsilon_{10}}(\ell)$	$\epsilon_{10} = \left(\frac{\ell}{11}\right)$						$A_{11}$
$\chi_3$	$SL_{10}(\ell)$	$\ell \equiv 1, 3 \pmod{8}$						$2.M_{12}$
$\chi_3$	$SU_{10}(\ell)$	$\ell \equiv 5, 7 \pmod{8}$						$2.M_{12}$
$\chi_4$	$SL_{10}(\ell)$	$\ell \equiv 1, 3 \pmod{8}$						$2.M_{12}$
$\chi_4$	$SU_{10}(\ell)$	$\ell \equiv 5, 7 \pmod{8}$						$2.M_{12}$
$\chi_5$	$\Omega_{11}(\ell)$	$\ell \neq 13$	$2^{10} \rtimes M_{11}$ $< 2^{10} \rtimes A_{11}$					$M_{12} < A_{12}$
$\chi_5$	$\Omega_{11}(13)$		$2^{10} \rtimes M_{11}$ $< 2^{10} \rtimes A_{11}$					$M_{12} < A_{12}$ $< A_{13}$
$\chi_6$	$SL_{16}(\ell)$	$\left(\frac{\ell}{11}\right) = 1$					$Spin_{10}^{\epsilon_{10}}(\ell)$	$M_{12}, 2.A_{11}$
$\chi_6$	$SU_{16}(\ell)$	$\left(\frac{\ell}{11}\right) = -1$					$Spin_{10}^{\epsilon_{10}}(\ell)$	$M_{12}, 2.A_{11}$
$\chi_7$	$SL_{16}(\ell)$	$\left(\frac{\ell}{11}\right) = 1$					$Spin_{10}^{\epsilon_{10}}(\ell)$	$M_{12}, 2.A_{11}$
$\chi_7$	$SU_{16}(\ell)$	$\left(\frac{\ell}{11}\right) = -1$					$Spin_{10}^{\epsilon_{10}}(\ell)$	$M_{12}, 2.A_{11}$
$\chi_8$	$\Omega_{44}^{\epsilon_{44}}(\ell)$	$\epsilon_{44} = \left(\frac{5}{\ell}\right)$						$A_{11}$
$\chi_9$	$P\Omega_{45}(\ell)$	$\ell \neq 7$					$P\Omega_{10}^+(\ell)$	$A_{11}, M_{12}$
$\chi_9$	$P\Omega_{45}(7)$						$P\Omega_{10}^+(7)$	$M_{12}, A_{11}$ $M_{23} < M_{24}$
$\chi_{10}$	$\Omega_{55}(\ell)$	$\ell \neq 13$	$2^{54} \rtimes M_{11}$ $< 2^{54} \rtimes A_{55}$				$\Omega_{11}(\ell)$	$M_{12} < A_{12}$
$\chi_{10}$	$\Omega_{55}(13)$		$2^{54} \rtimes M_{11}$ $< 2^{54} \rtimes A_{55}$				$\Omega_{11}(13)$	$M_{12} < A_{12}$ $< A_{13}$

TABLE 2. Obstructions to the maximality of  $M_{11}$  embeddings; the case  $\ell \neq 2, 3, 5, 11$

$\varphi$	Type of $X$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
$\varphi_2$	$\Omega_9(11)$						$A_{11}$
$\varphi_3$	$SL_{10}(11)$						$2.M_{12}$
$\varphi_4$	$SL_{10}(11)$						$2.M_{12}$
$\varphi_5$	$\Omega_{11}(11)$	$2^{10} \rtimes M_{11} < 2^{10} \rtimes A_{11}$					$M_{12} < A_{12}$
$\varphi_6$	$\Omega_{16}^+(11)$					$Spin_9^+(11)$	$2.A_{11}, M_{12}$
$\varphi_7$	$\Omega_{44}^+(11)$					$\Omega_9(11)$	$A_{11}$
$\varphi_8$	$\Omega_{55}(11)$	$2^{54} \rtimes M_{11} < 2^{54} \rtimes A_{55}$				$P\Omega_{11}(11)$	$M_{12} < A_{12}$

TABLE 3. Obstructions to the maximality of  $M_{11}$  embeddings; the case  $\ell = 11$

In column 2 of our tables we have recorded the classical group  $X$  for which  $H \in \mathcal{S}(X)$ , and in subsequent columns we record obstructions that we have found. The type of the obstruction is found in the column heading.

$\varphi$	Type of $X$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
$\varphi_2$	$\Omega_{10}^+(5)$						$A_{11}$
$\varphi_3$	$SU_{10}(5)$						$2.M_{12}$
$\varphi_4$	$SU_{10}(5)$						$2.M_{12}$
$\varphi_5$	$\Omega_{11}(5)$	$2^{10} \rtimes M_{11} < 2^{10} \rtimes A_{11}$					$M_{12} < A_{12}$
$\varphi_6$	$SL_{16}(5)$					$Spin_{10}^+(5)$	$2.A_{11}, M_{12}$
$\varphi_7$	$SL_{16}(5)$					$Spin_{10}^+(5)$	$2.A_{11}, M_{12}$
$\varphi_8$	$P\Omega_{45}(5)$					$P\Omega_{10}^+(5)$	$A_{11}, M_{12}$
$\varphi_9$	$\Omega_{55}(5)$	$2^{54} \rtimes M_{11} < 2^{54} \rtimes A_{55}$				$\Omega_{11}(5)$	$M_{12} < A_{12}$

TABLE 4. Obstructions to the maximality of  $M_{11}$  embeddings; the case  $\ell = 5$

$\varphi$	Type of $X$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
$\varphi_2$	$SL_5(3)$						
$\varphi_3$	$SL_5(3)$						
$\varphi_4$	$\Omega_{10}^+(3)$						$A_{11}, M_{12} < A_{12}$
$\varphi_5$	$SL_{10}(3)$					$SL_5(3)$	$2.M_{12}$
$\varphi_6$	$SL_{10}(3)$					$SL_5(3)$	$2.M_{12}$
$\varphi_7$	$\Omega_{24}^-(3)$					$PSL_5(3)$	
$\varphi_8$	$\Omega_{45}(3)$					$\Omega_{10}^+(3)$	$A_{11}, M_{12} < A_{12}$

TABLE 5. Obstructions to the maximality of  $M_{11}$  embeddings; the case  $\ell = 3$

$\varphi$	Type of $X$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{S}_{Lie}$	$\mathcal{S}_{altspor}$
$\varphi_2$	$\Omega_{10}^-(2)$						$A_{11}, M_{12} < A_{12}$
$\varphi_3$	$SU_{16}(2)$					$Spin_{10}^-(2)$	$A_{11}, M_{12} < A_{12}$
$\varphi_4$	$SU_{16}(2)$					$Spin_{10}^-(2)$	$A_{11}, M_{12} < A_{12}$
$\varphi_5$	$\Omega_{44}^-(2)$					$\Omega_{10}^-(2)$	$A_{11}, M_{12} < A_{12}$

TABLE 6. Obstructions to the maximality of  $M_{11}$  embeddings; the case  $\ell = 2$

We found the type  $\mathcal{S}$  obstructions via character restriction, using the character tables in GAP [13]. To determine fusion of conjugacy classes we used the function `PossibleClassFusions`. From the character degrees we can easily deduce that  $H$  possesses no obstructions of types  $\mathcal{C}_4$ ,  $\mathcal{C}_7$  and  $\mathcal{C}_6$ . The  $\mathcal{C}_2$  type obstructions were determined in [15].

We summarize our findings in the proposition below noting that the second claim is taken from Bray, Holt, Roney-Dougal [6].

**Proposition 3.1.** *If  $H \in \mathcal{S}(X)$  with  $H \cong M_{11}$ , then  $N_X(H)$  is maximal in  $X$  if and only if  $X^\infty = SL_5(3)$ . If  $X^\infty = SL_5(3)$  and  $X$  contains a graph automorphism, then  $X$  contains exactly one conjugacy class of subgroups isomorphic to  $H$ , else two.*

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